Lecture 07
Hypothesis Testing with Multivariate Regression

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Taylor B. Arnold
Yale Statistics
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Goals for today

1. Review of assumptions and properties of linear model
2. The multivariate T-test
3. Hypothesis test simulations
4. The multivariate F-test

Review from Last Time

Classical linear model assumptions

I. Linearity $Y = X\beta + \epsilon$

II. Strict exogeneity $\mathbb{E}\left(\epsilon|X\right) = 0$

III. No multicollinearity $\mathbb{P}\left[\text{rank}(X) = p\right] = 1$

IV. Spherical errors $\mathbb{V}\left(\epsilon|X\right) = \sigma^2\mathbb{I}_n$

V. Normality $\epsilon|X \sim \mathcal{N}(0, \sigma^2\mathbb{I}_n)$

Finite sample properties

Under assumptions I-III:

(A) $\mathbb{E}(\widehat{\beta}|X) = \beta$

Under assumptions I-IV:

(B) $\mathbb{V}(\widehat{\beta}|X) = \sigma^2(X^tX)^{-1}$

(C) $\widehat{\beta}$ is the best linear unbiased estimator (Gauss-Markov)

(D) $Cov(\widehat{\beta}, r|X) = 0$

(E) $\mathbb{E}(s^2|X) = \sigma^2$

Under assumptions I-V:

(F) $\widehat{\beta}$ achieves the Cramér–Rao lower bound

The T-test

Hypothesis tests

Consider testing the hypothesis $H_0 : \beta_j = b_j$.

Under assumptions I-V we have the following:

$$\widehat{\beta}_j - b_j \Big| X, H_0 \sim \mathcal{N}\left(0, \sigma^2 \left((X^t X)^{-1}_{jj}\right)\right)$$

Z-test

This suggests the following test statistic:

$$z = \frac{\widehat{\beta}_j - b_j}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}}$$

With,

$$z | \, X, H_0 \sim \mathcal{N}(0, 1)$$

T-test

As in the simple linear linear regression case, we generally need to estimate σ^2 with s^2.

This yields the following test statistic:

$$
\begin{aligned}
t &= \frac{\widehat{\beta}_j - b_j}{\sqrt{s^2\left((X^tX)^{-1}_{jj}\right)}} \\
&= \frac{\widehat{\beta}_j - b_j}{\text{S.E.}(\widehat{\beta}_j)}
\end{aligned}
$$

T-test

The test statistic has a T-distribution with $(n-p)$ degrees of freedom under the null hypothesis:

$$t|\, X, H_0 \sim t_{n-p}$$

This time around, we'll actually prove this.

To start, we re-write the test statistic as:

$$
\begin{aligned}
t &= \frac{\widehat{\beta} - b}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\
&= \frac{z}{\sqrt{s^2/\sigma^2}} \\
&= \frac{z}{\sqrt{q/(n-p)}}
\end{aligned}
$$

Where $q = r^t r/\sigma^2$.

We need to show that (1) $q|X \sim \chi^2_{n-p}$ and (2) $z \perp\!\!\!\perp q|X$.

Lemma If B is a symmetric idempotent matrix and $u \sim \mathcal{N}(0, \mathbb{I}_n)$, then $u^t B u \sim \chi^2_{\text{tr(B)}}$.

Proof: The symmetric matrix B can be written as by its eigen-decompostion; for some orthonormal Q matrix and diagonal matrix Λ:

$$B = Q^t \Lambda Q$$

Notice that as B is idempotent:

$$\begin{aligned}
(Q^t\Lambda Q) &= (Q^t\Lambda Q)(Q^t\Lambda Q) \\
&= Q^t\Lambda QQ^t\Lambda Q \\
&= Q^t\Lambda^2 Q
\end{aligned}$$

Since $Q^t = Q^{-1}$:

$$\Lambda = \Lambda^2$$

Therefore all the elements of Λ are 0 or 1.

By rotating the columns of Q, we can actually write:

$$\Lambda = \begin{pmatrix} \mathbb{I}_{\text{rank}(B)} & 0 \\ 0 & 0 \end{pmatrix}$$

Notice that this also shows that:

$$\begin{aligned} \text{tr}(B) &= \text{tr}(Q^t \Lambda Q) \\ &= \text{tr}(\Lambda Q Q^t) \\ &= \text{tr}(\Lambda) \\ &= \text{rank}(B) \end{aligned}$$

Now let $v = Qu$. We can see that the mean of v is zero:

$$\begin{aligned}\mathbb{E}v &= \mathbb{E}Qu \\ &= Q\mathbb{E}u \\ &= 0\end{aligned}$$

And the variance of v is:

$$\begin{aligned}\mathbb{E}vv^t &= Q\mathbb{E}(uu^t)Q \\ &= Q\mathbb{I}_n Q^t \\ &= \mathbb{I}_n\end{aligned}$$

Now we calculate the quadtratic form $u^t B u$ by the transform of v. Substitute that $u = (Q)^{-1} v = Q^t v$:

$$\begin{aligned}
u^t B u &= v^t Q B Q^t v \\
&= v^t Q (Q^t \Lambda Q) Q^t v \\
&= v^t Q Q^t \Lambda Q Q^t v \\
&= v^t \Lambda v \\
&= v^t \begin{pmatrix} \mathbb{I}_{\mathrm{tr}(B)} & 0 \\ 0 & 0 \end{pmatrix} v \\
&= \sum_{i=1}^{\mathrm{tr}(B)} v_i^2 \\
&\sim \chi^2_{\mathrm{tr}(B)}
\end{aligned}$$

Which completes the lemma.

(1) We know that $r^t r = \epsilon^t M \epsilon$, so:

$$\begin{aligned} q &= \frac{r^t r}{\sigma^2} \\ &= \frac{\epsilon^t}{\sigma} \cdot M \cdot \frac{\epsilon}{\sigma} \end{aligned}$$

Under Assumption V, $\frac{\epsilon}{\sigma} \sim \mathcal{N}(0, \mathbb{I}_n)$. Therefore from the lemma, then $q \sim \chi^2_{\mathrm{tr(M)}}$; last time we showed the tr(M) $= n - p$, which finishes the first part of the proof.

(2) The random variables $\widehat{\beta}$ and r are both linear combinations of ϵ, and are therefore jointly normally distributed. As they are uncorrelated (problem set 2), this implies that they are independent. Finally, this implies that $z = f(\widehat{\beta})$ and $q = g(r)$ and themselves independent.

So, therefore, we have:

$$\begin{aligned} t &= \frac{\widehat{\beta} - b}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\ &= \frac{z}{q/(n-p)} \\ &\sim t_{n-p} \end{aligned}$$

Simulation

F-Test

F-test

Now consider the hypothesis test $H_0 : D\beta = d$ for a matrix D with k rows and rank k.

We'll form the following test statistic:

$$F = \frac{(D\widehat{\beta} - d)^t [D(X^t X)^{-1} D^t]^{-1} (D\widehat{\beta} - d)/k}{s^2}$$

And prove that is has an F-distribution with k and $n - p$ degrees of freedom.

We can re-write the test statistics as:

$$F = \frac{w/k}{q/(n-p)}$$

Where:

$$\begin{aligned} w &= (D\widehat{\beta} - d)^t[\sigma^2 D(X^tX)^{-1}D^t]^{-1}(D\widehat{\beta} - d) \\ q &= r^t r/\sigma^2 \end{aligned}$$

We've already shown that $q|X \sim \chi^2_{n-p}$, and can see that $q \perp\!\!\!\perp w$ by the same argument as before. All that is left is to show that $w \sim \chi^2_k$.

Let $v = D\widehat{\beta} - d$. Under the null hypothesis $v = D(\widehat{\beta} - \beta)$. Therefore:

$$\begin{aligned} \mathbb{V}(v|X) &= \mathbb{V}(D(\widehat{\beta} - \beta)|X) \\ &= D\mathbb{V}(\widehat{\beta} - \beta|X)D^t \\ &= \sigma^2 D(X^tX)^{-1}D^t \end{aligned}$$

Now, for the multivariate normally distributed v with zero mean, we have:

$$w = v^t\mathbb{V}(v|X)^{-1}v$$

Now, decompose $\mathbb{V}(v|X)^{-1} = \Sigma^{-1}$ as $Q^t \Lambda Q$. Notice that this implies:

$$\begin{aligned}
\Sigma^{-1} &= Q^t \Lambda^{1/2} \Lambda^{1/2} Q \\
Q\Sigma^{-1} &= \Lambda^{1/2} \Lambda^{1/2} Q \\
\Lambda^{-1/2} Q\Sigma^{-1} &= \Lambda^{1/2} Q
\end{aligned}$$

From here, we can write w as the inner product of a suitably defined vector u:

$$\begin{aligned} w &= v^t \Sigma^{-1} v \\ &= v^t Q^t \Lambda^{1/2} \Lambda^{1/2} Q v \\ &= u^t u \end{aligned}$$

With a mean of zero:

$$\begin{aligned} \mathbb{E}(u|X) &= \mathbb{E}(\Lambda^{1/2} Q v | X) \\ &= \Lambda^{1/2} Q \mathbb{E}(v|X) \\ &= 0 \end{aligned}$$

And variance of

$$\begin{aligned}
\mathbb{E}(uu^t|X) &= \mathbb{E}(\Lambda^{1/2}Qvv^tQ^t\Lambda^{1/2}|X) \\
&= \Lambda^{1/2}Q\mathbb{E}(vv^t|X)Q^t\Lambda^{1/2} \\
&= \Lambda^{1/2}Q\Sigma Q^t\Lambda^{1/2} \\
&= \Lambda^{-1/2}Q\Sigma^{-1}\Sigma Q^t\Lambda^{1/2} \\
&= \Lambda^{-1/2}\Lambda^{1/2} \\
&= \mathbb{I}_k
\end{aligned}$$

And, finally:

$$w = uu^t \sim \chi^2_k$$

Which finishes the proof.

An alternative F-test

Now, consider the following estimator:

$$\widetilde{\beta} = \arg\min_{b} ||y - Xb||_2^2, \quad \text{s.t.} \quad Db = d$$

We then define the restricted residuals $\widetilde{r} = y - X\tilde{\beta}$.

An alternative F-test

An alternative expression for the F-test statistic is:

$$F = \frac{(\tilde{r}^t\tilde{r} - r^t r)/k}{r^t r/(n-p)}$$

Conceptually, it should make sense that this is large whenever the null hypothesis is false.

An alternative F-test

If we let SSR_U be the sum of squared residuals of the unrestricted model ($r^t r$) and SSR_R be the sum of squared residuals of the restricted model, then this can be re-written as:

$$F = \frac{(\text{SSR}_R - \text{SSR}_U)/k}{\text{SSR}_U/(n-p)}$$

This is the way it is written in the homework and in the Fumio Hayashi text. It is left for you to prove that this is equivalent to the other F test.

Application