Lecture o4 Applications and Intro to Multivariate Regression

14 September 2015

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Notes

- 1. Problem set 1 due start of next class
- 2. TA session tomorrow night
- 3. Two typo; 4(b) has hypothesis test for the intercept not the slope, 3(c) required a tweak to the probability
- 4. Try to get a fresh copy of notes!

Goals for today

- 1. Galton's heights data
- 2. Multivariate regression; normal equations
- 3. Model frames in R

Galton Heights Application

MULTIVARIATE REGRESSION MODELS

The multivariate linear regression model is, on the surface, only a slight generalization of the simple linear regression model:

$$y_i = x_{1,i}\beta_1 + x_{2,i}\beta_2 + \cdots + x_{1,p}\beta_p + \epsilon_i$$

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The statistical estimation problem now becomes one of estimating the *p* components of the multivariate vector β .

A sample can be re-written in terms of the vector x_i (the vector of covariates for a single observation):

$$y_i = x_i^t \beta + \epsilon_i$$

In matrix notation, we can write the linear model simultaneously for all observations:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{p,1} \\ x_{1,2} & \ddots & & x_{p,2} \\ \vdots & & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{p,n} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

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Note: we use the transpose for $x_i^t \beta$ but not for $X\beta$!

For reference, note the following equation

$$y = X\beta + \epsilon$$

Yields these dimensions:

 $y \in \mathbb{R}^n$ $X \in \mathbb{R}^{n \times p}$ $\beta \in \mathbb{R}^p$ $\epsilon \in \mathbb{R}^n$

Vector Norms

When working with vectors and matricies, it will be helpful to represent certain quantities by norms. The p-norm of a vector is given by:

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In particular, the squared 2-norm yields the sum of squares of a vector.

Vector Norm Properties

The following properties are true of all vector norms, for a scalar α and vectors v_1 and v_2 .

$$||\alpha v_1|| = |\alpha| \cdot ||v_1||$$
$$||v_1 + v_2|| \le ||v_1|| + ||v_2||$$

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Notice that the 2-norm is dual to itself.

p-Norm Properties, cont.

Hölder's inequality then yields

 $|v_1^t v_2| \le ||v_1||_p ||v_2||_q$

p-Norm Properties, cont.

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As a special case, the Cauchy–Schwarz inequality gives that:

 $|v_1^t v_2|^2 \le ||v_1||_2^2 ||v_2||_2^2$

p-Norm Properties, cont.

Finally, and of most importance for us today, note that the squared 2-norm is exactly equal to the self inner product:

$$||v_1||_2^2 = v_1^t v_1$$

Least squares (again)

To estimate the least squares solution, which is again the MLE for independent normal errors, we see that:

$$\widehat{eta} \in \mathop{\mathrm{arg\,min}}_{b \in \mathbb{R}^p} \left\{ || \mathbf{y} - \mathbf{X} eta ||_2^2
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Now using vector norms to denote the sum of squares.

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= $y^t Y - 2y^t X\beta + \beta^t X^t X\beta$

Normal Equations

In order to find the minimum of the sum of squares, we take the gradient with respect to β and set it equal to zero.

Recall that, for a vector *a* and symmetric matrix *A* :

 $abla_{eta} a^t eta = a$ $abla_{eta} eta^t A eta = 2A eta$

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This gives the gradient of the sum of squares as:

$$\nabla_{\beta} ||y - X\beta||_{2}^{2} = \nabla_{\beta} \left(y^{t}y - 2y^{t}X\beta + \beta^{t}X^{t}X\beta \right)$$
$$= 2X^{t}X\beta - 2X^{t}y$$

Setting this equal to zero gives a set of p equations called the normal equations:

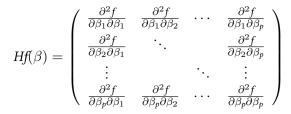
$$X^t X \widehat{\beta} = X^t y$$

Maximum or Minimum?

To determine whether the normal equations give a local minimum, maximum, or saddle point, we can calculate the Hessian matrix.

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If the Hessian is positive definite $(x^t H x \ge 0)$ at a critical point, then the critical point is a local minimum.

$$|\nabla_{\beta}||y - X\beta||_2^2 = 2X^t X\beta - 2X^t y$$

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We can see that the Hessian is simply:

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$$\begin{aligned} \mathbf{v}^{t}\left(2X^{t}X\right)\mathbf{v} &= 2\left(\mathbf{v}^{t}X^{t}X\mathbf{v}\right) \\ &= 2||X\mathbf{v}||_{2}^{2} \\ &\geq 0 \end{aligned}$$

Back to the normal equations themselves, notice that if the matrix X^tX is invertable, we can 'solve' the normal equations as:

$$X^t X \widehat{eta} = X^t y$$

 $\widehat{eta} = (X^t X)^{-1} X^t y$

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This is not a good way to solve the normal equations numerically, but for deriving theoretical results about the least squares estimator this form will be very useful.

MATRICIES AND MODEL FRAMES IN R